‘The So-Called Great Calculation According to the Indians’ of the monk Maximos Planoudes. A Translation

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Since numbers continue without bound, but knowledge of the boundless is not possible, the more eminent of the astronomers invented certain signs and a method relating to them, so that the representation of those numbers they needed might be more easily and more clearly apprehended at a glance. There are only nine signs required which are these: 1 2 3 4 5 6 7 8 9[1] They also use a certain other sign which they call a *cipher*, which, according to the Indians, signifies ‘nothing’. These nine signs are themselves of Indian origin and the cipher is written as 0.

When each of these 9 signs[2] stands alone by itself and in the first place beginning from the right-hand side, the symbol 1 indicates *one*, 2 indicates *two*, 3 *three*, 4 *four*, 5 *five*, 6 *six*, 7 *seven*, 8 *eight* and 9 *nine*. If, however, it is in the second place, then the symbol 1 indicates *ten*, 2 *twenty*, 3 *thirty* and so on. In the third position 1 indicates *a hundred*,[3] 2 *two hundred*, 3 *three hundred* and so on. The pattern continues for the remaining places.

[1]For ease of reading, I will use modern forms for the symbols.

[2]I will use numerals when they are given in the Greek and names when Greek names for numbers are given.

[3]Note that this is one word in Greek, as with the other multiples of a hundred. This will be important in understanding P.'s description of the four types later in the work.

Thus, in the first position the signs are to be regarded as units[4], which beginning from one proceed to nine. (Two, three, four, up to nine will be reckoned as *monadic* numbers, since they are all bounded by ten, neither reaching nor exceeding it.) Hence any sign which occurs in the first position will be regarded as *monadic*, and in the second position as *decadic*, that is, between ten and ninety, and in the third position as *hecatontadic*, that is, between a hundred and nine hundred. So also a sign in the fourth position is regarded as a multiple of a thousand and in the fifth as myriads [5] and in the sixth as tens of myriads, the seventh as hundreds of myriads, the eighth as thousands of myriads, and in the ninth as myriads of myriads. If one were to proceed even beyond this point, the tenth position counts as tens of myriads of myriads [6] and the eleventh as hundreds of myriads of myriads, the twelfth as thousands of myriads of myriads and the thirteenth as myriads of myriads of myriads. Indeed one could proceed even further.

Now to clearly illustrate what I have said by an example, suppose the given number is 8136274592 which occupies ten of these places.
We[7] begin from the right hand side, as has been said, and so the sign 2 in the first place indicates the number two, which is a monadic number. In the

[4]lit. 'monads'.
[5]Greek uses the word *μυριάδες* or 'myriads' for tens of thousands.
[6]It appears that the word *μυριάδων* has dropped out of the text here.
[7]I have everywhere replaced the first person singular with the first person plural.

second place, the sign 9, is ninety, which is a decadic number, consisting in fact only of tens, just as the two before it, being monadic, consisted only of units. The sign 5 in the third place is five hundred, which is a hecatontadic number. The number 4, in the fourth place, is four thousand, which is a *chiliadic* number[8], and 7 in the fifth represents seven myriads and is a *myriadic* number. The sign 2 in the sixth place is twenty myriads; it is a *decakismyriadic* number. 6 in the seventh place is six hundred myriads, a *hecatontakismyriadic* number. The sign 3 in the eighth is three thousand myriads, a *chiliontakismyriadic* number. The sign 1 in the ninth is a myriad of myriads, and is a *myriontakismyriadic* number. The sign 8 in the tenth place is eighty myriads of myriads, which is a *decakismyriontakismyriadic* number. The given number is thus read in its entirety as *eighty myriads of myriads and a myriad of myriads and three thousand six hundred and twenty seven myriads, four thousand five hundred and ninety two.*

For greater clarity let me also say the following: The number lying in the first place indicates the number of units there are. The second is the number of tens, the third the number of hundreds, the fourth the number of thousands and so on, as far as many places as the number occupies.

Note also that as the number proceeds through the four places it changes

[8] i.e. multiples of a thousand.

its literal name each time.[9] Then in the fifth place it takes again its original name, not just the number itself, but tied to its place value. This continues 'till the eighth place in which it takes the name of the fourth, and so it continues in turn. Thus, in the previous example given above, 2 indicates and is read as 'two', 9 as 'ninety', 5 as 'five hundred' and 4 as 'four thousand'. The sign 7 is then 'seven (units of) myriads', just as we referred to 'two' (units) in the first position. Similarly the next sign is for seven, but is in fact seven myriads[10], 2 is for twenty myriads, just as we had ninety in the second position, so here the sign means twenty, for both numbers are decadic. This is the same as the case with the monadic numbers that preceeded them, and so on in turn.

The cipher is never placed at the left-hand end of the digits but can appear in the middle of the number or at the right-hand side, that is, at the extreme side before the smallest (non-zero) place digit.[11] Not only one, but two, three, four or as many zeros as are required may be placed in the middle or in the other aforementioned place. Just as the (number of) places

[9]This and the following passage arise from a quirk of the Greek naming of numbers, rather than anything implicit in the number system. P. is saying that there are four types (lit. signs): units, tens, thousands and myriads. After this, we repeat the types as **units** of myriads, **tens** of myriads, **hundreds** of myriads and **thousands** of myriads. Then, we again repeat the types as **units** of myriads of myriads, **tens** of myriads of myriads and so on. The difficulty arises because Greek uses one word for say three thousand but two for three hundred thousand (three tens-of-myriads).

[10]lit. 'seven myriadically'.

[11]The Greek is verbose and clumsy, lit. 'it is placed at the extreme in the direction of the smallest numbers'.

increases the size of the number, so too the does the number of ciphers. For example, one cipher lying at the end makes the number decadic, 50 is fifty in fact, two ciphers make it hecatontadic, thus 400 is four hundred, and so on in turn. If one cipher lies in the middle and there is only one symbol before it, it makes that number hecatontadic, thus 302 is three hundred and two, but if there are two such signs, the number is chiliadic, thus 6005 is six thousand and five.

If there is a single cipher with two signs after it, this indicates a chiliadic number, thus 6043 is six thousand and forty three, but if there are two then the number is myriadic, thus 60043 is six myriads and forty three, and so on in turn. To put it simply[12], the number is to be understood by the order in which the symbols are placed.

In regard to astronomy we have then the need of six types of operations[13], of which the first is that referred to as numeration, [14] and then addition, subtraction, multiplication and division, while the sixth is the extraction of the square root[15] of each (given) number. There has already been mention made of the representations, so let us turn our attention to the method of addition.

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[12]Lit. 'To speak simply'. Wilson reads *'ειπεῖν* here for Gerhardt's *'ειπὼν*.

[13]Gk. *συμβαλλομένων*. Representing numbers is not strictly speaking an 'operation', but there is no simple English equivalent that captures the force of the Greek.

[14]Lit. 'the signs or schemata'.

[15]lit. 'taking the side of each number as though it were a square'.

ON ADDITION:

Addition is the combination of two or more numbers into a sum resulting in one number, for example when we add two and three we make five. It is performed as follows.

Write down the symbols in turn, as many and whichever ones you like, and again underneath these, write as many digits as you like, with the same number of places or more or less. Let them be placed so that the units[16] are placed under the units, the tens under the tens and so on in turn. Then add each to each, that is, the first to the first, the second to the second and so on in similar fashion. Write the number resulting from the two above the first column, that is, the number formed from adding the first column above the first column, that resulting from the second column above the second column and so on in turn. If then the total of the numbers added is two or three or four and so on as far as nine, it is written above, as has been said. But if the total is ten then write a cipher, which indicates 'nothing', take a one instead of ten and combine it with those numbers being next in line to be added. If the total is more than ten, write down that part of its excess over ten above the numbers added, as before, and take one instead of ten and combine it with those numbers about to be added.

[16] lit. 'monadics'.

To make the explanation clear by example, I give the following diagram.

8	0	3	0	2
5	6	8	7	8
2	3	4	3	3

We begin then with 3 and 7 ; 3 and 7 gives 10, write the cipher above, and carry[17] the unit, which indicates ten. Again, 4 and 8 gives 12, add in[18] also the unit which you carried and we get[19] 13. We again take the unit which indicates ten and write the 3 above. Now 3 and 6 gives 9, add in the unit you carried and we get 10. Write the cipher above and keep the unit; now say 2 and 5 makes 7, add in the unit you held and we get 8 and write this above. Thus the sum of five thousand six hundred and eighty seven and two thousand three hundred and forty three is eight thousand and thirty.

There follows a test[20] by which we can learn whether we have performed the addition correctly or not. Regard the values of the signs no longer as monadic, decadic and so on, but take them all as monadic and add (the numbers in) each of the rows individually and look at the number resulting from each. Subtract nine from each, look at what is left for each number, and place what is left at the end of the row which produced it. If the remain-

[17]lit. 'take hold of'.
[18]lit. 'take on board', a nautical term.
[19]lit. 'behold'.
[20]This is the old method of *casting out nines*.

ing numbers of the two rows add to less than nine, then it is not necessary to subtract (further) from them, but if what is left exceeds this, then again subtract nine and look at the remainder. If it equals the remaining number in the top third row, then one knows that the addition was effected correctly, but if they are not equal then the opposite is true. For the sake of clarity, let us demonstrate on the previous example.

We say that 8 and 3 is 11, take way 9 leaves [2. Now 5 and 6 is 11, take away 9 leaves 2, 2 and 8 is 10, take away 9 leaves][21] 1. And 1 and 7 is 8 and so write 8 in turn in the middle row. Again 2 and 3 is 5, 5 and 4 is 9, take away 9 and the resulting number 3 remains. Write this in turn after the numbers added together. Now 3 and 8 is 11, take away 9 and 2 is left over. You then have the number equal to the number from the previous check and so the addition is correct.

So much for addition.

ON SUBTRACTION or TAKING AWAY:

Subtraction from a number, means to take one number from another and look at what is left over. We always either take the lesser number from the greater, so that there is some remainder, for example three from five leaves two, or we subtract equal from equal, where the remainder is nothing; for

[21] This section in square brackets has been restored to the text by Gerhardt.

example when we subtract three from three. It is not possible to take a greater from a lesser number, for it is not possible to take away what is not there.

If you now want to perform a subtraction, you proceed as follows:
Write the digits in turn as many and whichever you like. Below it write the same number of digits or less, but not more. If the number below which is being subtracted has fewer digits then no other condition need be stated. But if it has the same number of digits as the top number then make sure the last digit of the top row is greater than the last corresponding digit of the bottom row. As stated previously, when I speak of the 'last digit', I mean the one on our left hand side. We need this digit to be greater than the bottom one to ensure that the whole number (on top) is greater than the whole number (on the bottom). This being so, even if the rest of the digits in the bottom number are greater than the digits in the top number, taken one at a time, nevertheless the top number is greater and in no way smaller than the bottom one, that is, than the number being subtracted. Suppose then that the numbers are written with digits corresponding, units with units, tens with tens and so on in turn. If the first digit[22] on the bottom row is less than the first digit of the top, then take the lesser from the greater and write the number remaining above the first digit on the first row. If it is equal,

[22]Gk. Here the word *σημεῖον* 'sign' is used instead of the the usual *σχῆμα*. I have translated both as 'digit'.

then subtract equal from equal and, as stated above, write the remaining zero above. If it is greater, since we cannot take the greater from the lesser, borrow a unit which signifies ten from the next digit after it, that is from the second digit of the bottom row. (This digit after the first place occupies the second decadic position.) Having added this ten to the smaller number in the top row, subtract the larger from that total and write the remainder again above the smaller digit. If you can take the second digit on the bottom line, after adding the unit to it - for this unit is again thought of as a unit in respect to the number in the corresponding column, but as ten in respect to the preceeding column, since every number is taken to be ten times the number in the preceeding column, viz; ten compared to the unit, a hundred to ten, a thousand to a hundred and so on - if then you can take away this second digit, after adding to it the unit, then take it away and write the remainder, if there is any, above that second digit on the top row, but if not then write 0. Again, if the second digit on the bottom row, with the unit, is greater than the second digit on the top, again borrow a unit, that is a ten, from the third column on the bottom and add the ten to the second digit on the top and take away the second digit on the bottom, with the unit, and again write the remainder above the second digit on the top. Proceeding thus to the end, you will have the desired result. For the number remaining after you take the whole of the lesser number from the whole of the greater is precisely the number written above the top row.

So that what is being said might become clearer to you in an example, let it be as follows:[23]

I wish to take 3 from 2 but I can't, since 3 is greater than 2. I add a unit to the 4 next to the 3. I regard the unit as a ten[24] and I say 10 and 2 (make 12). I subtract 3 from 12 leaving 9. I write this number above the 2. Again I wish to take 4, along with the unit, from 1 , but am unable. I add a unit to the 8 after the 4 and, regarding this as ten, I say ten and 1 is 11.

5	4	6	1	2
1	8	7	6	9
5	4	6	1	2
3	5	8	4	3
1	1	1	1	

I take from 11 the 4 with the unit, that is 5, leaving 6 and I write this above the 1. Using this method I reach the last column and since I can subtract 3 with a unit from 5, I subtract from it 4, that is 3 with a unit, leaving a unit. This number I write above the 5. With this in mind another example is given, in order to show that if the remainder is zero, it is still written. Beginning in the third place in the accompanying example,

[23]We are here subtracting 35843 from 54612 giving 18888769. The carried units are written underneath and the larger number is written again above, which will be used as part of the checking process later.

[24]Literally ' as decadic'

3	1	2	5	6
0	7	0	8	8
3	1	2	5	6
2	4	1	6	8
1	1	1	1	

after adding the unit to the digit thus making 2, I can subtract this completely from 2 and nothing is left. I write 0 above the two. I continue in the same way to the end of the diagram.

A further example will show how the calculation is carried out when there are fewer digits on the bottom line.

6	4	5	4	3	2
6	3	8	6	7	4
6	4	5	4	3	2 [25]
		6	7	5	8
	1	1	1	1	1

Going to the 4th place, since I am unable to take the 6 with the unit added, that is 7, from the 5 I write a unit next to the units under the 4 and I regard this as 10. Having added this to the 5, I proceed as previously explained. After this I take the unit from the 4 leaving 3 and I write this above the 4. If there are two, three or more digits less in the bottom than the top[26] a unit is not written below those digits from there to the end along the top row, but the digits are written as they are, each one written above itself in turn after

[25] The first 1 on the bottom line appears to have dropped out of the text.

[26] Read *‘ο κατωτέρω στίχος τοῦ ’ανωτέρω*, ... instead of *‘ο κατωτέρω στὶχος, τῶι ’ανωτὲρωι*.

those digits written above and included with the remainders.[27]

If you wish to have a check, proceed as follows. Suppose in the last example, we say 8 and 4 is 12 and take the unit from it leaving 2. Write this above the 4. Again, 5 and 7 is 12, add in the unit you carried and we see 13, take away 10 and again carry the unit leaving 3 . Write this above the 7. Again 7 and 6 is 13, adding in the unit makes 14. Then take away 10 and carry the unit leaving 4. Write this above the 6. Again 6 and 8 is 14 and adding in the unit makes 15. Take 10 amd again carry the unit leaving 5. Write this above the 8. Then since we have nothing further from the bottom line of our initial two rows, take the next digit in turn of those left, that is 3, and again add in the unit which you held making 4. Write this above the 3. Now since 6 has nothing to combine with[28] nor is there a unit being carried to add to it, place this 6 as it is above itself. Now the topmost row turns out to be the same as the top row of the two initial rows, and the subtraction was performed correctly. Indeed this is always the method by which the check is made, that is, to add the lower of the two initial rows to the row of remainders. Check that the row resulting from this is the same as the top row of the initial two. If so, then the subtraction has been correctly done.

[27] This almost incomprehensible sentence seems to be simply saying that when we have noting to subtract from a digit, we just copy the digit down.

[28] Read ‘*ενωθῶσιν* instead of ’*ενωθῶσιν*.

There is also another method of subtraction effected as follows:

I place two rows of numbers having either the same or an unequal number of digits. Suppose they are unequal then it is clear that the top has more digits than the bottom, and should they be equal then let the last digit on the top be greater than that of the bottom as prescribed. If then, the first digit on the bottom is either equal to or less than the first digit on the top, then it is subtracted from it as previously made clear. But if it is greater, I borrow a unit for that lesser digit from the second digit after it on the top row, which becomes a ten as it moves into the first position. The second digit is then reduced by a unit, since a unit has been removed from it. For example, if it were 4 it becomes 3, and so I write 3 above the 4. I add ten to the smaller number and I subtract from this sum the greater number and I write the remainder above the smaller number. If I can subtract the second digit in the bottom row from the second digit in the top, I subtract that given number no longer from 4 but from 3, but if not then I again borrow a unit from the third to the second digit and proceed as previously explained. If the bottom digit completely cancels with the top digit then I write 0 above the top one.

Let us make our explanation clear by example.[29]

0	8	9	8	4
2	4	0	3	1
3	5	1	4	2
2	6	1	5	8

Let it be as shown. I wish to take 8 from 2 but this is not possible. I borrow a unit for the 2 from the 4 and the 4 then becomes 3 which I also write above the 4 and add the unit, or rather the ten, to the 2 to make 12. I subtract the 8 from this leaving 4 which I also write above the 2. Again, I wish to take 5 no longer from 4 but from 3 but I can't. I borrow a unit for the 3 from the unit in the third place, the remainder being 0 which I write above this unit. Adding the unit to 3 I make 13 and I subtract from it

5[30] leaving 8. This I write above the three. In similar fashion I wish to take 1 from 0 but this is impossible. Again I borrow a unit for the 0 from the 5 and I continue with the above procedure. If the top has either one or two zeros through the middle of the line again I make use of borrowing, and if there is a single zero I proceed as in the accompanying example.

0	6	7	8	8
2	9	1	3	
3	0	2	4	6
2	3	4	5	8

[29]We are subtracting here 26158 from 35142.
[30]Reading ω for ω

Moving to the third place, since I am unable to take 4 from 1, I wish to borrow a unit for the 1 from the 4th place, but I find that there is nothing in the fourth place. I proceed then to the fifth and take a unit, which becomes a ten when moved to the 4th place. But since the numbers, that is to say the digits, never exceed nine, I again take a unit from the ten and I borrow this for the unit in the third place. It becomes 11 and so the subtraction proceeds. This is the situation in the case of a single zero. If there are two, I proceed as in the next example.

0	6	7	7	9
2	9	9	1	
3	0	0	2	4
2	3	2	4	5

Moving to the second place, since I am unable to take 4 from 1, I wish to borrow a unit for the 1 from the third place, but in the third place I find zero. I move then to the fourth place and I also find a zero in it. I continue to the fifth place and take a unit from it, which, when moved to the fourth place, becomes ten. Since nine is left over in the fourth place, I again take a unit from the ten in third place and I borrow from it a unit for the 1 in the second place. Using the nine that is left in the third place, I continue the subtraction following the previous instructions. The method for checking in the method is the same as it was in the previous method.

ON MULTIPLICATION:

Multiplication takes place when we take one number a certain number of times, and from this produce a new number. For example, twice three is six. We took three, as many times as there are units in two, or vice versa, which produces six. Multiplication is performed as follows.

Write down the symbols in turn, as many and whichever ones you like, and again underneath these, another row of digits, with the same number of places or more or less. Do this however in an orderly fashion so that unit is under unit, the tens under the tens and so on in turn. Then multiply the first digit on the top row with the first digit of the bottom row. If the resulting number is less than ten[31] then write it above the first, but if a multiple of ten[32], such as ten or twenty or thirty and so on, write zero[33] and carry as many units as there are tens in the product, but if (the product) is made up of both kinds, that is, units and tens, such as fifteen or twenty four and the like, then write the number of units above; for example five or four, but in regard to the tens, carry as many units as there were tens. Now multiply the first digit of the top line with the second digit of the bottom line and again the first digit of the bottom line with the second digit of the top and take the

[31] lit. *monadic*

[32] lit. *decadic*

[33] lit. *nothing*

sum of these products. Add on the monads which you carried, whether two or more, and again, if a number less than 10 results, write it down above the second digit, or if a multiple of ten or mixed, proceed as previously shown. Now if each row has only two digits, there remains only to multiply the second digit by the second digit and then add to this any units that might be carried and write the result after the numbers previously written. If there are three digits, multiply the first by the third and again the first by the third chiastically[34] and also the second by the second and adding these up write it down as you were instructed.[35]. Then multiply the second by the third and the second by the third chiastically, record it and then multiply the third by the third and again record it.

To make the explanation clear by example, I give firstly the following diagram having two rows each with two digits.

8	4	0
	2	4
	3	5

We say four-times five is twenty and write zero[36] above the 4 since twenty is a multiple of ten, and carry the two. We then say four-times three is twelve and five-times two is ten and together they make twenty two. To this add the two units we carried and it becomes 24. We write the 4 above the 2 and carry the two units. Again we say twice three is six and add on the two units

[34]that is, in cross formation like the Greek letter chi χ

[35]i.e. as well of course as adding on any units that were carried

[36]lit. nothing.

and it becomes eight. I then write 8 in turn next to the previous digits and this number is the product of twenty four by thirty five.

Now consider another diagram where the lines have three digits.

1	1	4	0	4	8
			4	3	2
			2	6	4

We then say twice 4 is eight and write this above the 2. Twice 6 is 12 [and four-times 3 is 12][37] and together they make 24. We write 4 above the 3 and carry 2. Again twice 2 is 4 four-times 4 is 16 and thrice 6 is 18 and together they make 38. We add the two units to this to give 40. Write zero above the 4 and carry 4. Then we say thrice 2 is 6, six-times 4 is 24, total 30. Add on 4 gives 34. Write 4 next to the zero and carry 3. Now we say four-times 2 is 8 and add 3 making 11 and write this in turn after the 4.

This is how the multiplication proceeds if there is an equal number of digits in each of the rows, but if one exceeds the other, fill up the row with the smaller number of digits with zeros and repeat the method outlined. To make this clear by example, we illustrate as follows:

7	6	8	4	2
	1	4	2	3
	0	0	5	4

Thrice 4 is 12. Write 3 above the 2 and carry 1. Also thrice 5 is 15 and four-times 2 is 8 together giving 23. We add on the unit, total 24 and write 4 above the 2 and carry 2. Now thrice zero is zero, four-times 4 is 16, twice

[37]This section in square brackets has been restored to the text by Gerhardt.

5 is 10 , a total of 26. Add on the two giving 28. We write the 8 above the 4 and carry 2. Also thrice zero is zero, four-times 1 is 4, twice nothing is nothing and five-times 4 is 20 giving 24 . Add on 2 to give 26 . Write 6 above the the 1 and carry 2. Now twice nothing is nothing, five-times 1 is 5, four-times 0 is 0 making 5, and add on 2 gives 7. We write this in turn next to the 6. Then four-times 0 is 0, nothing-times 1 is nothing and (so) we do not write anything, also once 0 is 0 and again I do not write anything.

Observe that when you have multiplied the first digit (on the top row) by the last digit (on the bottom row), you should make some sign on the first digit to indicate that it has been multiplied by all the digits in turn and must not be multiplied again.

When performed on a larger example[38], the process of multiplication might be summarised as follows: Multiply the first digit by the first digit and write it down, then the first by the second and the first by the second chiastically and write it[39]. Then the first by the third and the first by the third chiastically and the second by the second[40] directly[41] and write it. Then the first by the fourth and the first by the 4th and further the second by the third

[38]In what follows Planudes is thinking of multiplying two five digit numbers

[39]In all that follows, the instructions to add the products as we multiply, carry and add the previous carry are all to be understood.

[40]Reading *'έτι τὸν* $\overline{\beta^{ov}}$ *'επὶ τὸν* for *'έτιτον* $\overline{\beta^{ov}}$ *'επτιὸν*.

[41]Greek *πρὸς 'ορθὰς*. Wäschke observes that this has the opposite meaning to *chiastically* and that numbers which stand underneath each other are always multiplied directly.

and the 2nd by the 3rd chiastically and write it. Then multiply the first by the fifth and the 1st by the 5th and again the 2nd by the 4th and the 2nd by the 4th chiastically and further still, the third by the third directly and write it. Now make some mark over the first and multiply the 2nd by the 5th, for it (the second) has already been multiplied by the 3rd and 4th, and the second by the 5th (chiastically) and further the third by the 4th and the 3rd by the 4th and write it. Now make some mark over the 2nd. Take the 3rd by the 5th and the 3rd by the 5th, for it (the third) has become the second, and also the 4th by the 4th directly and write it. Then, having put a mark against the 3rd, take the 4th by the 5th and the 4th by the 5th and write it. Finally take the 5th by the 5th and write it.

To begin with then, we multiply the first digit by the first digit, then by the second digit, but proceed no further with this since there are no digits between the 1st and the 2nd. Then once again, the first by the third and since the second is between the first and the third, the second is multiplied by itself. Then multiply (the first) by the 4th and then since the 2nd and 3rd are between the 4th and 1st, the 2nd is multiplied by the 3rd. Next (the first) is multiplied by the 5th and since the 2nd, 3rd and 4th are between the 1st and 5th, the numbers at each extreme are multiplied, that is, the 2nd and the 4th, then the middle one, here the 3rd, is multiplied by itself. In a similar way, if the first were multiplied by the sixth, then the second, 3rd, 4th and 5th would be between the 1st and the sixth. Thus first of all, the numbers at each extreme are multiplied together, that is, the second and the fifth, then the middle ones, the third and fourth, each pair being combined chiastically. You can continue the same process as far as you like. After the first digit is multiplied by all the digits, then likewise the second and the third and the fourth and the remaining digits are multiplied (by the last digit) through to the end, each exactly once, and all the numbers in between are multiplied in the manner described above. Finally the last is multiplied by itself which brings the process to completion.

You should also observe that when the first (digit) is multiplied by the first, a single multiplication is performed, for the first digit is multiplied only by the first. When, however, it is multiplied by the second, it is a double multiplication, the first by the second and again the first by the second. When multiplied by the third a triple multiplication occurs, namely the 1st by the 3rd and the 1st[42] by the 3rd as well as the second by itself, and so on. To put it simply, the number of multiplications is indicated by that digit which the first one multiplies, for example, the first required a single multiplication, the second double, the third triple and so on. The type of multiplication corresponds to the position of the digit. Now when the multiplication begins not from the first but from the second or third digit and so on to the end, for we always multiply to the end as we explained before, then there are as many

[42] reading $\overline{\alpha o\varsigma}$ for $\alpha o\varsigma$

multiplications as there are digits. So when multiplying the second, we count from the second or from the 3rd when we begin at the third and so on for each digit to the end, for the number of multiplications is indicated not from the position of the digit, as happened when we started from the first, but by the number of places (from digit we multiply by). There is also here a rather elegant and natural aspect to observe. The number of multiplications begins at one and returns to one after reaching the number of digits in each row, if the rows have an equal number of digits or the greater number if unequal. In other words the number increases from one and finally returns to one[43]. Let me clarify this by example. Suppose each row, or the larger row, has three digits. The first by the first is a single multiplication. The first by the 2nd and again the first by the second is a double multiplication. The first by the third [and again the first by the third][44] and further the second by itself is a triple multiplication. The 2nd by the 3rd and the second by the third is double. The third by the third is simple. You see then how it proceeds from single to double then to triple and descends one by one from triple to double and then to single.

The following is a test of the correctness of the multiplication. Take the digits of the numbers set out in the two rows and add separately the digits of each. As in the test for addition, we regard the value of each digit as

[43]lit. *until returning to one it stays there*

[44]This appears to have fallen out of the text and was restored by Gerhardt.

monadic. Cancelling off nine from the total, write separately the remaining number for each row. Now multiply these together and if the product is less than 9, record it, or if greater than 9 cancel off 9 and take the remainder. Repeat the process on the third row obtained from the multiplication of the two earlier rows and similarly take the remainder. If it less than 9, look at the remainder and if it is equal to the remainder obtained from the other two rows then you can conclude that the multiplication was correctly performed. Here is an example.

2	9	4	8	4	9	9
			5	4	3	3
			5	4	3	3

The addition of the row resulting from the multiplication, viewing the digits monadically, is 36. Cancelling of the 9's, 9 is left. The remaining two rows each have sum 12 and cancelling off 9 from both, each has three left. Three times three is 9.

I do not think it would be excessive[45] to present another method of multiplication. This method is rather difficult to perform using paper and ink, but, since it requires numbers to be erased[46] and others written in their place, should be done on a slate sprinkled with sand. Written in ink, this would lead to a great deal of confusion, but on sand it is easy to erase some numbers with your finger and write other numbers in their place. Writing in the sand

[45]Reading *περιττόν* for *εριττόν*
[46]reading '*εξαλείφειν* for '*εξαλείφειν*

is not only preferential in multiplication but also in all the other methods we have discussed and in those yet to be mentioned. This method is as follows: Take two rows of numbers with as many digits of whatever type you like. Place them so that the first digit of the top row lies under the last digit of the top row. Multiply the last digit of the top row by the last digit of the bottom row and, if less than ten, write the resulting number above the last digit of the bottom row but on the same line as the top row. If (the product) is a multiple of ten, write zero above that last digit. Write 1 if the product is ten, 2 if twenty and so on, next to the zero on the left hand side. If the product is a mixture of units and tens, then write the units in the space above and then the tens next to that number you have just written, as explained above. Now multiply that same last digit of the top row by the second last one on the bottom row and write the resulting number, if less than ten, above (the second last digit of the bottom row) as described. If the product is greater than ten, then take the number of tens, treat them as units and write the sum of this and the number written above the last digit (of the bottom line) above that last digit. You will need to erase that last digit first. Having done this and having multiplied the last digit of the first row each time by the digit after the one just multiplied, proceed to the first digit of the bottom row and multiply it also by that same last digit of the top row. Now erase the last digit of the top row[47] and write the resulting product above the first

[47]This will of course destroy part of the number we are multiplying by. When this rather exotic method is completed, the top number in the multiplication will be completely lost.

digit of the bottom row. We now erase all the bottom row and transpose it one place directly to the right of the position it previously occupied so that the first digit is under the second last digit of the (original) top row,[48] the second digit where the first was[49], and the third where the second was, to as many places as there are. Now continue multiplying the second digit from the end of the (original) first row by each of the digits in the bottom row, following the steps outlined above. Again transpose the numbers (in the bottom row) until the first number of the bottom row coincides with that of the top row, and when the multiplication is completed you will have the final product neatly and correctly written down.

Let us clarify what has been said by an example and let the two rows be as shown.

		6	5	4
6	5	4		

We say six-times 6 is 36 , write 6 above the (bottom) 6 and 3 in turn next to that 6. Again we say six-times 5 is 30 and write 0 above the 5. Now 30 is a multiple of ten so I take 3 instead of 30 and combine this with the 6 that is next to the 0 making 9. Now erase that 6 and write 9. Again we say six-times four is 24 . Write 4 above the 4 after first erasing the 6 at the end of the top row and instead of 20 take two units and write 2 erasing the zero.

[48]That is, the digit next to the last digit of the original number which we erased.

[49]reading '$\tilde{\eta}\nu$ for '$\tilde{\eta}\iota\nu$

The resulting number is shown:

3	9	2	4	5	4
	6	5	4		

Now I transpose the bottom row giving the following arrangement.

3	9	2	4	5	4
		6	5	4	

Now we say five-times 6 is 30 . We do not write anything above the 6. There is a 2 there and so there is no need to make any further change. Now in place of 30 take 3 units and combine them with the 9 immediately after the 2 giving 12, from which I erase the 9 and replace it with 2, take a single unit and add it to the 3, which is immediately next to the 2 we have just written, to get 4, and write that 4 in place of the 3 which is erased. Again we say five-times 5 is 25 . From this we add the 5 to the number (four) immediately after the 5 to make 9 which we write after erasing the 4. Now take two units from the 20 and add them to the number 2 written after the 9 to make 4, erase the 2 and write 4. Again we say five-times 4 is 20 . Now erase the 5 and above the 4 write 0. Instead of 20 take two units and combine them with the number 9 after the 0 just written to make 11 . From this write 1 above the 5 and cross out the 9. In place of ten take one unit and add it to the 4 after the 5 you have just written. Erasing the 4 and writing 5, the sum will as appear as follows:

4	2	5	1	0	4
		6	5	4	

Then once again transpose the bottom row so the arrangement becomes:

4	2	5	1	0	4
			6	5	4

Again we say four-times 6 is 24 . Add the 4 to the 1 above the 6 giving 5. Erase 1 and write 5. Take two units from the 20 and add them to the 5 after the 5 just written to make 7. Erase 5 and write 7. Again we say four-times 5 is 20 . Now since there is a zero above the 5, we do not write anything, but from the 20 we take two units and add them to the 5 next to the 0 giving 7, so we erase 5 and write 7. Again we say four-times 4 is 16 . Cross out the 4 and write 6 above the 4 and from the 10 we take one unit and cross out 0, writing 1 next to the 6. The sum then becomes:

4	2	7	7	1	6
			6	5	4

As you can see, the multiplication is complete. The test for correctness by this method is the same as for the previous method of multiplication, that is we compare the product[50] of the first two rows with that of the 3rd row. In the diagrammatic presentation of the example, we have left out some of the steps, since it is impossible using ink to arrange the digits under one another. Note that the digits in the top row are altered during the multiplication process, while those in the bottom line are not. Also note that whenever any digit on the top row is multiplied by the digit directly under it in the bottom row, this digit in the top row is not added to the product, but is erased and

[50]by which he means the product of the rows after 'casting out nines'.

only the number obtained from the product is written down. When however, a digit is multiplied by one in a different position, the resulting number is added to the digit above the one which is multiplied.
So much for multiplication, let us now consider division.

ON DIVISION:

Division takes place, when dividing one number into another we see how many multiples of the one there are in the given number we are dividing into. For example, when six is divided by three, we see how many multiples of three there are, and there are two, since three times two is six. Division can occur either from a smaller number by a greater one, [51] or between two equal numbers, or from a larger by a smaller. A smaller number is divided by a larger one as follows: write the smaller one above and the larger one below. If the one above is three and the lower one twelve, we say that the number of twelves in three is three twelfths, which is one quarter. If three is above and five below we say that it is three fifths and so on in turn.
The lesser number is divided by the larger one as follows: Suppose we wish to divide three by five. [52] The dividend ought to be greater than the divisor, for thus it would be made up of a certain number of multiples of the divisor,

[51] Lit *one with greater appearance*, *πρόσωπα*.

[52] Lit. a third by a fifth. This section is unnecessary, given his earlier comments. Planudes divides 3 into 15 equal parts (each a fifth) and divides this by 5 giving 3. He then gives a tedious digression on cancellation of fractions.

but here the dividend is smaller. We need some method whereby we can make the three greater than the five and then we would be able to perform the division. This is done as follows: I divide each unit of 3 into as many parts as the divisor 5 has units. Since there are three units in three, each divided into five, this clearly produces 15. This fifteen is fifths of units, since units divided into five produce fifths, just as six produces sixths and so on. 15 then, in this multiple counting,[53] is clearly greater than 5. Hence three, the smaller number, cut into parts commensurate with the greater, becomes greater than the greater. Once this is done, the division is now clear. Divide the 15 by 5 producing 3, just as if it had remained undivided, and so we get three fifths for each unit of 5, since the fifteen units were fifths. In this manner the lesser is divided by the greater, by taking the number of parts equal to the smaller, but the denominator determined by the greater. In our example there were three parts, this was the lesser but the fractions were fifths, taking their denomination from the greater, 5. There are situations when we can condense the such a fraction to a single unitary fraction [54], for example let us write three twelfths as one quarter. But there are situations when we can't; three fifths, for example, which cannot be contracted to a unitary fraction. This is dealt with in the following way: We must examine the two numbers - I refer to the lesser and the greater - to see if they have some common factor except one, that is, either two or three and so on, and

[53] 'απάριθμησιν.

[54] Lit. condense the fraction to one part and name it with one name.

look for the largest factor. For example 8 and 12 have a common factor of two but also of four. Thus in fact, four is the (greatest) common factor of the two numbers, not 2. If then the numbers have a common factor, we can simplify the fraction [55] and make its real nature [56] more comprehensible. For example, divide 4 by 20. Dividing by 20 we obtain four twentieths[57]. Now since four is itself a common factor of 4 and 20, the four goes into itself once, but into 20 five times, so we say that 4 twentieths is one fifth, 'one' from the 'once' and the 'fifth' from the 'five times'. You should think of the 'fifth' as the fifth part of unity, that is, as a fifth part of a unit of 4, or as follows: Since 4 goes into 20 five times, we divide each unit of it into 5 and it becomes 20 fifths. Now associate to each unit, one of the 20 fifths. We count through all the numbers from the first in multiples. Let us speak further about this. Suppose we divide 8 by 12. We say then: Associate 8 twelfths to each unit of 12. Since four is a common factor of 8 and 12, that is, four goes twice into 8 and three times into 12, we say 8 twelfths is two thirds, that is two thirds[58] of a unit, 'two' because of the 'twice' and 'third' because of the 'three times'. The numerator takes it name from the smaller number, but the denominator from the [larger][59] quotient, thus 'three' for the 'three times'. Hence this is way we do it in the case when the numbers have a common factor, but if

[55] Lit. name their parts with a smaller name.

[56] Gk. '*υπόστασις*.

[57] Lit. associate to each unit of 20 four twentieths.

[58] The first mention of 'two thirds' is *δύο..τρίτα* while the second is *δίμοιρον*.

[59] Not in the Greek.

their common factor is one, then the fraction remains as it was, for example, three divided by 5 becomes three fifths and this cannot be reduced[60], but retains its form.

Now it is important to realise that whether we are dividing the lesser number by the larger, or whether the larger by the smaller, the numerator of the fraction we obtain from division will always be the same as the number being divided and the denominator the same as the number by which we are dividing. For example, suppose we divide 4 by 12, the lesser by the greater, we say. Associate to each unit of 12, 4 twelfths which is a third of a unit. The 'four' is the same as the number we are dividing into and the 'twelfths' from the name of the number we are dividing by. Again, suppose 12 is divided by 4, the greater by the lesser. We associate to each unit of 12, a quarter, which gives three units. Here also the numerator of the fraction is 12, the same as the 12 being divided, whereas the denominator, that is the 'four', is the same as the 'four' which we are dividing by. Observe how the 4 twelfths is one third of a unit, but the twelve quarters gives three units. In general, if the lesser is divided by the greater, it produces fractions of unity and if greater by lesser, then units. In conclusion, this is the method of greater numbers into lesser numbers, and the division between two equal numbers is abundantly clear, each unit of one matches with each unit of the other, but division of

[60] Gk: *συστεῖλαι*.

smaller numbers into larger numbers is multifaceted. Either the division is by a *monadic number*[61], that is from one to nine, or by a *decadic*[62] number, that is, from ten or twenty in turn through ninety, or by some combination of these, or by a *hecatonadic*[63] or by a *hecatonadic* together with a *decadic* or by a *hecatonadic* together with a *decadic* and a *monadic* or even beyond as far you might wish to proceed. First of all, let us describe how division by a *monadic* number is performed:

Let a row of however many digits you like be set down and then under it place the *monadic* number, so that between it and the given number there is room to place another row. If the last digit[64] of the row is greater than the *monadic* number see how often the smaller digit is able to be subtracted from the larger one, and write the number of such possible subtractions below the larger number. If, after subtraction of the smaller from the greater as often as it can be done, nothing remains of the larger, again subtract this same *monadic* number from the penultimate digit and as often as you subtract this *monadic* number from it, again write this below it. But if there is some remainder from the greater, write this in small print above, between the last and penultimate digit, and combine it with the penultimate digit. Regard this remainder as a *decadic* number and the penultimate digit as a *monadic* number and again see how many times you can take the *monadic* number

[61]i.e. single digit.
[62]i.e. multiple of 10.
[63]i.e. multiples of 100.
[64]i.e. the last digit counting from the left.

from this mixed number. Again write the number of such subtractions below the penultimate digit. Continue this process until you come to the first digit[65]. This is the procedure if the last digit is greater than the *monadic* number, however, if it is equal to it, take the *monadic* number from it once and write one below the last digit, then move to the penultimate digit and proceed according to the above instructions. Now if the *monadic* number is greater than the last digit, resulting in an inability to take the larger from the smaller even once, then combine the last and penultimate digits and, making them a mixed number, subtract the *monadic* number from it as often as you can, and write the number of such subtractions under[66] the penultimate digit rather than the last one. You can then proceed further. Suppose we draw a diagram in which the *monadic* number is smaller than the first digit.[67]

	1			
4		8	6	5
1		6	2	1
3				

I can take 3 once from 4 and I write 1 below the 4. One unit is left. I place this in small print above, between the last and penultimate digits and combine it with the penultimate digit 8 to produce 18. Now since I can again subtract 3 from 18 six times, I write 6 below the 8. There is, in this case, no remainder. Again I can subtract 3 twice from the six and I write 2 below the 6, and there is no remainder. Again[68] I can subtract 3 once from 5 and

[65]counting from the left.
[66]The Greek has ’ή which Wäschke reads as ’ήτοι.
[67]We are here dividing 4865 by 3.
[68]Reading πάλιν for πάν.

I write 1 below the 5 and from the 5, 2 remains. I write this outside of the row[69] and I divide each of the two units, and more if there were any, each into as many units there are in the *monadic* number. There were three of these so I divide two by three and I say that a third of 4865 is one thousand six hundred and twenty one and two thirds, that is, two lots of one third. Let us now draw a diagram in which the *monadic* number is equal to the last digit. I then say 4 into 4 goes once and write 1 below the 4. Twice 4 is 8, and write 2 below the 8. Once 4 is 4, so I write 1 below the 8 leaving 2. As previously stated, I write this in small print above between the 6 and the 5.

			2	
4	8	6		5
1	2	1		6
4				

Combining this with the 5 I get 25 and taking the 4 from it six times I write 6 below the 5. One unit is left giving one quarter. So one quarter of 4865 is 1200 and sixteen and one quarter. In general then, the larger number here is divided by the smaller, in particular 4865 by 4, and (each digit) in turn by equal and smaller and larger; by equal when 4 was divided into 4, by the smaller, when 8, 6 or 5 were divided by that same 4 and by the greater when the unit was divided by the 4.

Again, let us set forth a diagram where the *monadic* number is greater than the first digit.[70]

[69] Although he says this, the diagram does not contain this digit.
[70] We are dividing 4865 by 5.

		3	1
4	8	6	5
	9	7	3
5			

Since we cannot take 5 from 4, I combine the 4 and the 8 to make 48. I subtract from it nine times 5 which is 45 and I write the 9 under the 8 leaving 3. Combining this with the 6 gives 36. Subtract from this seven times 5 and write the 7 under the 6 leaving one unit. This I combine with the 5 making 15 and I take three times 5 from it, leaving no remainder and I write the 3 under the 5. This gives one fifth of 4000 eight hundred and 65 to be 973.

This then is the situation when the number by which the row is divided is *monadic*. If it were *decadic* and we set down a row which is *decadic*, then I take the bottom from the top, it alway being necessary for the number at the top to be greater than the lower one. Having subtracted it as often as possible, the number of such subtractions I write below the top number. If there is no remainder I continue as has been explained above. Consider the following diagram: I take the 3 once from the 4 and write 1 below the 4, leaving one. I divide this by 30 and say that one thirtieth of forty is one unit and one thirtieth.

	1
4	0
1	
3	0

In the case when the number above is *decadic* and the one below is *monadic*, we do the following: I take 3 from 4 once and write 1, a unit remains. This, with the remaining naught, makes ten, and I take from it the 3 three times and I write 3 below the 0, leaving one. I divide this by 3 and I say that one third of forty is 13 and one unit.[71]

	1	
4		0
1		3
3		

Now if the top number is *decadic* and the bottom one is mixed, then I preceed as follows:[72]

	1	
4		0
1		
3		6

I take the 3 once from the 4 and I write 1, leaving 1. This becomes 10 and I take 6 from this once leaving 4 and I say that a thirty sixth of 40 is one unit and four thirty sixths, which is $\frac{1}{9}$. This is the situation then if the upper number is *decadic*. If it is mixed we proceed as follows:[73]

		13		16		1		17		
	2		3		2		1		3	
8		5		6		9		7		8
3		5		7		0		7		
2		4								

I place the prescribed line of digits and below it the mixed number which is 24. I say then, four times 2 is 8. I want to take four times 4 from 5 but I

[71] i.e. one unit of thirds.
[72] The accompanying diagram is not in the original.
[73] Here we are dividing 856978 by 24.

can't. I therefore take one unit from the four[74] and I say three times 2 is 6. I write the 3 below the 8 leaving 2. I write this in small print in the space above midway between the 8 and the 5 and again I say three times 4 is twelve. We had 2 and 5 which is 25 and I take from it the 12 leaving 13[75] . I write this directly above the 5. I do not write anything under the 5 because this same 3 which lies under the 8 is to be multiplied by the 2 and the 4. Again I can take 2 from the 13 six times, leaving 1 which is combined with the 6 and becomes 16.[76] But since I cannot take six times the 4 from 16, I take a unit from the six[77] and I say five times 2 is 10. I write 5 below the 5 and carry 3 which I write in the space above between the 5 and the 6 and I say five times 4 is 20 leaving 16. This I write directly above the 6. Since I cannot subtract 24 once from 16 (I say[78] eight times 2 is 16 but I cannot take eight times 4 from the 9 so I write 7 and say seven times 2 is 14 carry 2 which I combine with the 9 giving 29) and I say seven times 4 is 28 leaving 1.[79] Since I cannot take the 2 from this, I combine it with the 7 to make 17. (Since[80] I again cannot subtract 24 from this, I write 0 and carry 17). I can take the 2 from the 17 eight times, but since I cannot take eight times the 4 from the 18, the remaining unit combined with the 8, thus I take one unit from the

[74] Lit. four times.
[75] i.e. $85 - 3 \times 24$.
[76] i.e. $136 - 5 \times 24$.
[77] Lit. Six times.
[78] The section in round brackets is a restoration. There is a lacuna in the text.
[79] i.e. $169 - 2 \times 24$.
[80] The section in round brackets is not in the Greek.

eight[81] and I say seven times 2 is 14 and write the 7 below the 7 remainder 3. This I place beside the 8 and make 38 and I say seven times 4 is 28 leaving 10 which I write outside of the line. I do not write anything below the 9 but leave it as nothing[82] since from it we could not take the last digit of the mixed number which was always taken first, that is, the 2. Nor do we write anything under the 8 which I say is on the right hand side for the same reason.

We should note the following points: When the number is divided by a *monadic* number we subtract this *monadic* number from the larger number as often as it goes. When we divide by a mixed number, we do not always see how many times the last digit[83] goes, which we always take first, (although there are times when we do see it), as was demonstrated in the above diagram when we could have said four times 2, but we said three times[84] 2. This will happen when we can take the last digit from the last number but not be able to take the first digit from the second last. It is important to realise that the first and last digits were not taken from the same number, nor the last from the last and first from the third last, but if we take the last from the last it is also necessary to take the first from the second last. In fact the opposite can happen in the first and the last could be subtracted from the same number, as happened with the 25 in the previous diagram, for we subtracted three

[81] Lit. eight times.

[82] In fact the manuscript has a 0.

[83] Counting from the right.

[84] Reading $\tau\rho\acute{\iota}\varsigma$ for $\tau\rho\epsilon\acute{\iota}\varsigma$.

times 4 and five times 2 and further that the first digit was taken from a certain number, but the last from three digits along from it, as if fact in the previous diagram, the first was taken once from the end, but the last from the third from the penultimate, when two zeros fell in turn between them. It should be clear to you, whether to place nothing (or 0) or perhaps one or two or more, (that is, the number of times you took the last digit from the given number), under the number. We explained that the symbols placed under the number are to be understood as *monadic* rather than *decadic*, as in the previous diagram[85] after 2 and 7 and 2. Since I cannot take the 7, which is the last digit of the two by which the division is being done, from the 2, I combine the 2 with the 7 and produce 27. Thus three times 7 is subtracted from 27, indeed, from all of the 27 since three times 7 could not be subtracted from 2 but from the total 27. For this reason I put the 3 under the 7. If it were not[86] 7 then then 2 would not become twenty. We would then not only have a nought standing in the middle, but also have one at the end, by which I mean the right hand end. Now if two spots or places are left over, under which nothing has been written, under the left hand one I write nought, but under the right hand one, nothing at all. If one place is left, then I place nothing at all under it. To see when to write nought consider the following diagram[87]:

[85]This cannot refer to the previous diagram. Possibly another example has fallen out of the text. This whole section is very difficult to follow and very obscure.

[86]Not in the Greek.

[87]There is no diagram in the text.

	2		6		2		
		2		6		2	
2	9		4		5		3
	1		9		0		
2	7						

I say once 2 is 2, and once 7 is 7 and then I could also say once 2 is 2, but since I cannot take 7 from 4 once, and I say nine times[88] 2 is 18 and with the 3 makes 23. Outside of the line, I write the remainder[89] which is 23 out of 27, which is what I was dividing by. The spaces under the 5 and the 3 remain empty, so I place a nought under the single space below the 5, but write nothing at all under the 3.

ON THE CIRCLE OF THE ZODIAC:

Let us now make some comments about the zodiac, degrees, first parts, and secondary parts.[90] Note then that the sun passes through the 12 signs of the Zodiac as it passes through its cycle. Each of these is divided into 30 degrees and each degree is divided into 60 primary parts and each primary part into 60 secondary parts, and each secondary part into 60 tertiary parts and so on forever. Astronomers ignore all the rest and only concern themselves with 4 quantities, that is, the signs of the Zodiac, degrees, primary parts, which they simply call 'minutes' and 'seconds' leaving out the mention of 'parts'

[88] Little of this makes any sense and there appears to be a lacuna.

[89] Reading *λοιπά* for *λεπτά*.

[90] Greek: *μοίρων* -divisions, *λεπτῶν πρώτων* - primary parts = minutes, (*λεπτῶν*) *δευτέρων* - secondary parts= seconds.

from the 'seconds' and 'primary' from the 'minutes'.

ON ADDITION:

Whenever you wish to perform an addition, you do it as follows: Write down zodiac, degrees, minutes and seconds, whatever you find in the ****[91] of Astronomy, but if not, then whatever you like. Below write the number of the zodiac, the number of degrees in their columns and likewise the number of minutes and seconds. For example, the following diagram makes clear what we are saying.

Zodiac	Degrees	Minutes	Seconds
0	22	57	35
3	18	44	52
4	23	54	49
2	25	32	25
1	14	45	29
2	22	12	2

So beginning from the last[92] entry, 9 and 5 make 14 and 9 is 23 and 2 is 25. Write the 5 above and hold the 2 which we will add to the next column. 2 and 2 is 4 and 2 is 6 and 4 is 10 and 5 is 15. See how many times we need to subtract 6 from 15 and we see it is 2 remainder 3. Write the 3 above the second row, above the 5 and the 2 under the 5 in the minutes column. Say 2 and 5 is 7 and 2 is 9 and 4 is 13 and 4 is 17. Write the 7 above the four

[91] The Greek word here is illegible and not included in the text. Possibly it is some word for *data*

[92] lit. *smallest*.

and carry one. Again we say 1 and 4 is 5, 5 and 3 is 8 and 5 is 13 and 4 is 17. Check again how many times we can take away 6. It is 2 remainder 5. Write the 5 and carry the 2. Take this with the degrees and say 2 and 4 is 6, 6 and 5 is 11 and 3 is 14 and 8 is 22. Write the 2 and carry the 2. Say 2 and 1 is 3 and 2 is 5 and 2 is 7 and 1 is 8. See how many times you can take 3 from 8 and it goes 2 remainder 2. Write the 2 and carry 2, which is twice thirty. Again say 2 and 1 is 3 and 2 is 5 and 4 is 9 and 2 is 12. Take away twelve and zero is left.[93] The answer is 22 degrees, 57 minutes and 35 seconds and the sign of Ares[94].

ON SUBTRACTION:

Subtraction is performed as follows. Consider the diagram and begin from the last number. We say 4 from 5 leaves 1. Write this above.

0	27	30	41
10	24	58	25
9	27	27	44

It is not possible to take 4 from 2. Borrow a unit for the 2 from the 58. This unit represents 60 and so 57 is left. Instead of 60 in fact we take the 6 and say 6 and 2 is 8, take away the 4 and 4 is left and write this above the 2. Now we say 7 from 7 is zero, (for we previously took away a unit), and write this down. Again we say 2 from 5 leaves 3 and write it. Again we cannot

[93]Recall that 12 signs is 360°.

[94]lit. *first constellation of the ram*, Greek κριοῦ.

take 7 from 4 and so we borrow a unit for the 4, representing 10, and say 7 from 14 leaves 7. Now take the 1, which you borrowed, and combine it with the lower 2 giving 3 and say 3 from 2. This is not possible, so we borrow for it a one from the 10, that is 30 degrees,[95] and regard it as a 3, so we say 3 and 2 is 5. Take from this a 3 leaving 2 and write it. Now say 9 from 9, for we took away 1, is zero and write it.

It is clear that whenever you borrow a unit either from the minutes for the seconds or from the degrees for the minutes, then the unit represents 60, but from the zodiac numbers for the degrees, it represents 30. However when the borrowing is from degree for the degrees, or minutes for minutes or seconds for seconds then it represents 10. Now to perform a check, add the result of the subtraction to the smaller array so as to obtain the larger. Thus take 4 and 1 to get 5. If the sum of the two digits is beyond six, you should subtract 6 and hold the remainder. If the sum is a plain 6, then write zero and understand a one instead of a six. In regard to degrees, you have to subtract three if the sum exceeds this, and carry one instead of three and write down the remainder. In regard to zodiac numbers, if the total exceeds twelve, subtract twelve and write what is left. In the case when you cannot take a zodiac number from the zodiac number above it, then you borrow 12, and just as you did in the case of degrees, minutes and seconds, in one

[95] One twelfth of a revolution.

case borrowing 3 for 30 and in the other 6 for 60, so you subtract and write down the remainder. If you wish to subtract zodiac numbers, degrees, minutes and seconds from zodiac numbers and degrees, then write the zodiac numbers and degrees followed by two zeros on the top line and underneath the zodiac numbers, degrees, minutes and seconds.

Consider the following diagram:

1	24	5	15
10	24	0	0
8	29	54	45

Subtract one from the 24 degrees and borrow this for the zero in the minutes column. Regard this 1 as 60, since a degree is divided into 60 minutes. From this in turn, we borrow a one, which indicated a 60, for the zero which is in the seconds column, and the minutes and seconds then are clearly 59 and 60 respectively. Now we say 45 from 60 leaves 15, write this. Again [54 from 59 leaves 5, write this and say][96] 9 from 3, since you borrowed the 1 for the zero next to it. This cannot be done, so borrow a one for it, which indicates 10. 10 and 3 then is 13, take away 9 and 4 remains. Write this. Then combine[97] the 1 which you borrowed and say 1 and 2 is 3. 3 from 2 doesn't go, so borrow 30, that is one zodiac number, which is 3 and say 3 and 2 is 5. Take the 3 from the 5 and write 2. Since we took away a 1, we say 8 from 9

[96] This has fallen out of the text.

[97] Instead of taking 1 from the two at the top, he adds the borrowed 1 to the 2 at the bottom.

leaves 1 and write it. If you wish to take zodiac numbers, degrees, minutes and seconds from zodiac numbers alone, then write zeros in the columns for degrees minutes and seconds next to the zodiac numbers, and borrow one from the zodiac numbers, which represents thirty degrees, for the degrees, and again borrow from the degrees one, which represents sixty minutes, for the minutes, thereby leaving 59 degrees and write 60 in the minutes column. Again we borrow from this one, which signifies 60 seconds for the seconds, and this leaves 59 minutes, and (write) 60 in the seconds column. Subtraction is performed then, according to the method we have outlined. So much for subtraction.

ON MULTIPLICATION:

We will now speak about multiplication. We observe that whenever we multiply a degree by a degree, degrees are produced. When degrees are multiplied by minutes then minutes are produced, (when degrees are multiplied by) seconds then seconds, by tertiary parts, then teriary parts and so on. When we multiply minutes with minutes they produce seconds, and with seconds, they produce tertiary parts. Seconds by seconds produce quarter units since two by two is four. Whenever you want to multiply degrees and minutes by degrees and minutes, then multiply as shown in the following diagram:[98]

[98]We are here multiplying $8°16'$ by $14°13'$. In minutes, that is: $496 \times 863 = 428048$. Dividing by 3600 we obtain 118, which is $3 \times 30 + 28$, with a remainder of $3248 = 54 \times 60 + 8$.

3	28	54	8
	14	23	
	8	16	

We say then, ten-times 8 is 80, four-times 8 is 32. 80 and 32 are 112. Again ten-times 10 is 100, four-times 6 is 24 and ten-times 6 is 60 and four-times 10 is 40. Together these make [224. Furthermore twenty-times 8 is 160 and three-times 8 is 24. Together they make][99] 184. Now 224 and 184 make 408. Again twenty-times 10 is

200, three-times 10 is 30, twenty-times 6 is 120, and six-times 3 is 18 and together they made 368. Now see how many times sixty may be taken from this and you have 6 minutes with 8 seconds remaining. Write this and carry across the 6 minutes to the 408 minutes and you have 414. Again see how many times you can take away 60. It goes 6 times leaving 54. Write this down and combine the 6 degrees with the 112 degrees resulting in 118. Divide this into lots of 30 and you have 3, with 28 degrees remaining. This gives 3 (signs of the) zodiac, 28 degrees, 54 minutes and 8 seconds.

ON DIVISION:

When dividing degrees, minutes and seconds into degree minutes and seconds you proceed as follows. Reduce the numbers to the one unit[100] as shown in the accompanying diagram.

[99] The text in square brackets has been added by Gerhardt.
[100] Greek: *γένος*

3	23	54	
2	34	24	
	12234		1
	9264		
	2970		
	178200		19
	9264		
	2184		
	131040		14
	9264		

In the example given, say three times sixty is 180 and you have converted 3 degrees into minutes. Combine this with the 23 minutes. Then convert all the minutes into seconds and combine these with the seconds. To be clear on how this is done look at the accompanying diagram. Write the 23 minutes below and the 180 above and add them by saying 3 and 0 is 3, 2 and 8 is 10, write down the 0 and carry 1. Combine this and say 1 and 1 is 2. Write this and multiply by 60. Say three times 6 is 18, write the 8 and carry one. Further, six-times zero is 0, so write the one which you carried. Again six-times 2 is 12. Write this down and add to it the 54 seconds. Using this same method repeat the process on the bottom line, that is, convert the units to seconds and divide the seconds by seconds, giving degrees, since multiplying seconds by degrees produces seconds. Write down the resulting number of degrees and resolve the remaining seconds into tertiary parts. For example, in the accompanying diagram, the second row 2 34 24 resolves into seconds as 9264. I divide the greater by the lesser, that is, seconds by seconds, and the division gives 1 unit with remainder 2970. I claim that this produces one degree for indeed seconds were divided by seconds. Seconds cannot be multiplied by anything except degrees to produce seconds. I multiply [101] the remaining 2970 again by 60 to produce 178200. I divide again by the lesser of the two rows of numbers, that is, the third row by the second and after the division I have nineteen minutes. Seconds cannot be multiplied by anything except minutes to produce tertiary parts and so tertiary parts divided by seconds produce minutes. I then write down 19 and I claim this to be in minutes. When nineteen times the lesser row is subtracted from 189200, then 2184 tertiary units are left, and again I resolve these into quarter parts giving 131040. Again I divide the quarter parts by the oft mentioned (number in) seconds and 14 results from the division. I claim that these are seconds and write them. I maintain these are seconds since seconds cannot be multiplied by anything except seconds to produce quarter parts and indeed quarter parts divided by seconds produce seconds. Using this method I can proceed further and resolve quarter parts into fifth parts. Dividing these by seconds I produce tertiary parts, since tertiary parts times seconds produce fifth parts. Again the remaining fifth parts I resolve into sixth parts and divide these by seconds to give fourth parts since fourth parts time seconds produce sixth parts. I continue this as far as I wish. It is easiest to take the division as far as seconds, rather artificial to proceed as far as sixth parts and totally needless and excessive to go beyond that.

[101] Lit. *resolve*

Take note that when we resolve degrees into minutes, we perform a multiplication by sixty, but when we resolve minutes into seconds and seconds into tertiary parts and so on, we do not always multiply by sixty but sometimes by six. In the present example, so as to avoid any confusion by dropping out zeros, we resolved everything using sixty. But when multiplication by six is involved, where a 0 stands on the end, nothing has dropped out and when there are two zeros we have one[102]. I should also add that we regard the smaller number to be the minutes, the greater to be the seconds, the next greater the tertiary parts and so on. You could also say the opposite is true then minutes are greater than seconds, which are in turn greater than tertiary parts and so on. Whenever the greater number[103] is divided by the lesser one must look to see what number multiplied by the lesser produces the greater one and identify it as the result of the division. For example, to divide seconds by sixth parts, I look for the unit which when multiplied by seconds produces sixth parts and I discover it to be fourth parts. I then say that sixth parts divided by seconds makes fourth parts. Whenever equal (units) are divided by equals then degrees are produced. For whenever a number is multiplied by degrees, the type of unit is preserved. For example to divide fourth parts by fourth parts, I see what one multiplies by fourth

[102] This passage is obscure.

[103] Planudes is referring here to the fact that there are 60 seconds in a minute and hence he says that a minute is greater than a second. He often uses the word *number* when he means *unit*.

parts to preserve again fourth parts, and I find that it is degrees. Hence I find that degrees result from the division. Similarly, every unit multiplied by degrees remains the same. This is the reason that we say degrees are lesser than every other unit, whether minutes or seconds and so on. For example when I divide tertiary parts by degrees, I look to see what multiplied by the smaller, that is degrees, produces tertiary parts, and I find it to be tertiary parts and so the result of dividing tertiary parts by degrees is tertiary parts. There are other things one can say as well. Whenever you divide a greater unit by a lesser one, take the smaller from the larger and call the number that remains after the division, the remainder. For example, if you wish to divide third parts by seconds, take two from three then one is left. We claim that the result from the division is minutes. When you divide equal by equal, we must revert back to degrees, which results from the division, since taking one number from the other nothing is left. For example, subtracting third parts from third parts, we say that degrees result. Whenever you divide a number by degrees, [the unit remains the same] since degrees do not count as a number, being smaller than every other unit. For the units[104] of minutes, seconds and so on are like the constituents of the natural numbers. For example, when you divide third parts by degrees, it is not necessary for the unit resulting from the division to become seconds, but they remain third parts, for no number is subtracted since it is only a degree which is not a

[104]Lit. *numbers*

number but only has the notion[105] of a number.

ON FINDING THE SQUARE ROOT OF ANY NUMBER:

Since we have dealt with different problems arising from astronomical calculation, let us now deal with the quadrature of numbers which are not perfect squares, to show clearly that it is possible to find the square root of any given non-square number. This is achieved in the following way.

Take the square root of the nearest[106] perfect square and double it. Then take from the number whose square root you seek, the square you found nearest to it, and express the remainder as a fraction of the number obtained by doubling the square root[107] of the square.[108] For example, if eight were double the side of the square, take the fraction as eighths, if ten then as tenths, and so on.

Thus, suppose we wish to regard 18 as a square and find its root. Take a root of the perfect square nearest it, that is, of 16; its root is 4. Double this and obtain 8, and subtract 16 from 18 leaving 2, express this in terms of eighths and write that the square root of 18 is 4 and two eighths. Now

[105]Greek: $\lambda\acute{o}\gamma o\varsigma$
[106]i.e. nearest but less than
[107]lit. *side*.
[108]Symbolically, $\sqrt{a^2 + \epsilon} \approx a + \frac{\epsilon}{2a}$.

two eighths equals a quarter and so the root is 4 and a quarter. To see that this in the correct answer, multiply 4 and a quarter by itself and so find the answer is 18. Here are the steps: We say then that four-times 4 is 16 and four-times two eighths, that is, one quarter four times, equals four quarters equals one. Furthermore [four-times][109] two eighths is one, and one and one is two. Combine this with the 16 and it makes 18. But this method is too simplistic and is incomplete, and it has a lack of accuracy, for this answer is not the root multiplied by itself. If we multiply also a quarter times itself, we get not just 18, but 18 and one sixteenth. In what follows a more accurate method will be outlined, which we claim to be our own discovery, with the help of God. Let us turn our attention to this method for a while, showing how it can be used even for large numbers, so that it might become easier for us to understand. We apply the aforementioned method to numbers from the unit to 99 and from a hundred to 9999, since the roots of all numbers between these two have two digits, and we should proceed by the following method.

Suppose we write the number 235 and seek to find its square root.

	1		3		
2		3		5	
1	5				10
	2			10	

Take then the number which when multiplied by itself exactly produces the first digit two or is nearest to it. 2 is not correct since when multiplied by

[109]This seems to have dropped out of the Greek, restored by Gerhardt.

itself it gives 4. The number produced by the multiplication must be either equal to 2 or less than it. Hence the product should be one times itself, and we say one times one is again one. Subtract this from 2 and one is left over. Write this in small print above in the space between the 2 and the 3, as you did in the section on division. Now double the *one* which you found from the subtraction of one from 2. I refer here to the *one* which was the root not the square, for you should double that number which you found from the subtraction[110]. One and one is two. Write this *two* below the 3, but not so that it lies on the same line as the unit previously written below the 2, but below that again in the third row. Now find the number which when multiplied by this cancels with 13, but when multiplied by itself cancels with the residual[111]. I mean that the numbers must be less than or equal to 13 and the residual. This is what is always meant by 'cancel'.[112] Six multiplied by two can be subtracted from 13, but when multiplied by itself produces a number which is greater than the residual. The remainder is 13[113] and the square is 36. Hence we pass over the 6 and take 5 and say five-times 2 is 10.[114] Write the 5 between the 3 and the 2 on the same row as the unit written below the 2. Taking the 10 from the 13 leaves 3. Write this in small

[110]This is confusing. He means that you double the root, not the remainder.

[111]i.e. 35.

[112]Greek ἀναίρεσις.

[113]The text has 15 which I do not understand.

[114]Planudes has made the algorithm very complicated by not using place value. An easier may to express what he is doing is the say that we seek the largest integer x so that $(20 + x) \times x < 135$. Here $x = 6$ doesn't work, but $x = 5$ does.

print above in the space between the 3 and the 5. Again we multiply the 5 by itself giving 25 and subtract from 35 leaves 10.[115] Write this outside of the row by itself. Now double the5 and it becomes 2, just as at the beginning we doubled the unit below the 2. Write this in the third row in turn next to the two you wrote previously. Now combine the 20 and the 10 in the third row, since 2 lies in the tens column, this gives 30. Take half from it and we get 15, since 30 is double the root. The root then of 235 is 15 and ten thirtieths. The remainder is expressed in terms of that same thirtieth which is obtained by doubling the (integer part) of the square root. Now ten thirtieths is a third. Note that the (integer part) of the square root, that is 15, lies in the second row and its double is in the third. To check the result, proceed as follows: Multiply 15 by itself and by ten thirtieths and say, fifteen by 15 is 225, then fifteen times a third, that is ten thirtieths, gives 5 then 5 and 5 is 10. Combining this with 225 gives 235.

Now look further to the case when the square root has three digits, so the number is between 10000 and 999999. The problem is to find the square root of 421354.[116]

[115] i.e. $135 - 5 \times 25 = 10$.

[116] The Greek has 12, in place of 13 in the top row of the table, and although this is followed by Wäschke I cannot see that it is correct. This number is a remainder which is given in the text as 13.Wäschke also has 9 intead of 5 in the second row which is clearly a misprint.)

	6	13	117	9	23	
4	2	1	3	5	4	
	6	4	9			153
		12	8	18		

I seek a number which when multiplied by itself will cancel not just with the 4, but which is either less or equal to 42.[117] I find this number to be 6. Subtracting 36 leaves 6. I write this six below that 2 for we subtracted not (only) from the 4 but from the 42 taken together. The remainder 6 (I place above) between the 2 and the 1. Doubling the 6 lying below the 2 makes 12 which I write under the 1 in the third row as indicated. I then see what number multiplied by 12 cancels with the 61 and I find the answer is 5. [[118] Now since 5 multiplied by itself cannot be subtracted] from the following 13, I pass over the 5, take the 4 and say fourtimes 12 is 48 remainder 13. I write the 4 between the 1 and the 12 in the second line. You need to place it directly among the numerals in the second row under the number from which it was cancelled. Now combining the remaining 13 with the 3 we have 133. I take from this 4 multiplied by itself, that is 16, leaving 117.[119] I write this directly above the three and double the 4 to give 8 and I write this[120] after the 12 on the 3rd row. I now see what number multiplied by 12 cancels firstly

[117]This is explained by Planudes *infra*, in a (typically) long winded fashion. He could have simply pointed out in the beginning that we count off the digits in pairs from the right to the left. This may, as in the first example, leave a single digit on the left.

[118]There is a lacuna in the text at this point. I follow the reconstruction of the sense from Wäschke's translation.

[119]So $61-12\times 4 = 13$, and $113-4\times 4 = 117$. It is much easier simply to say $613-124\times 4 = 117$.

[120]Reading $\tau\alpha\tilde{\upsilon}\tau\alpha$ for $\tau\alpha\tilde{\upsilon}\alpha$.

with 117, then by 8 (cancels with) the next number and further multiplied by itself, cancels with the final part.[121] I find this number to be 9 and nine times 12 is108, leaving 9. This is write in the line above the 3 and the 9[122] and the 9 which I multiplied by the 12, I write between the 3 and the 8. Again I say nine times 8 is 72 leaving 23. I combine this with the 4 and it becomes 234. Again I say nine times 9 is 81. I subtract this from the 234 leaving 153 [123] and I write this by itself outside of the first row and I double the 9 giving 18. I write this in turn next to the 8 in the third row. The square root of 421394 is thus found to be 649 and one hundred and fifty three over one thousand two hundred and ninety eight, since 1298 is twice 649. Sometimes we cancel the number multiplied by itself with the number lying completely alone at the beginning, and sometimes we use it and the number after it[124]. We will briefly explain this and also what number we multiply by each time.[125]

If the number to be square-rooted has an odd number of digits, then you take only the first digit, but if there is an even number of digits, you take the second combined with the first. Also observe that if the number whose root you seek has one or two digits, then the root has one digit. If the number has three or four digits then the root has two, if five or six then three, if seven or eight then four and so it continues. To find the number of digits in the

[121] This is equivalent to finding the largest integer x such that $(1280 + x) \times x < 11754$.

[122] The Greek has 'between the 3 and the 9'.

[123] That is, $11754 - 1289 \times 9 = 153$.

[124] In the above example, he worked with 42 rather than simply 4, but in the first example, he worked with the first digit 2.

[125] See footnote 117

number you must double, or nearly double the number of digits in the root. Now no-one will blame me if I say that this method is my own invention, that is, how to find the square root of a number when that root has four digits. Numbers whose roots have four digits range from 1000000 to 100000000 just as those with five digit roots range from 100000000 to 10000000000 and those with six from 10 billion to 10 trillion[126] and so on according to the number of digits written above. Now we will discuss a number whose root has four digits, using the method which we have discovered.

Suppose then that we want to find the root of 16900963 .

			1 1	9 1	8	7				
1	6	9	0	0	9	6	3			
	4	1	1		1			6	4	2
		8	2	2	2					

Beginning in the same manner as the previous method, we say four-times 4 is 16 and I write the 4 below the 6. I double this and obtain 8. I write this and subtract it once from the 9. I write this one [127] leaving one. I write this [128] and it becomes 10. I subtract the one times itself from the 10 leaving 9 and write it. I double the one and it becomes 2. I again subtract the 8 from the 9 once, write the one and carry one. I write this and it becomes 10. I subtract the 2 from this once and get 8, I write this and further subtract

[126] Written in the Greek as $\ddot{\rho}$ to $\ddot{\alpha}$.
[127] Lit. 'the once'.
[128] Understand 'above'.

the one multiplied by itself leaving 7. Up until the 7 we have the root 411. I now look for a digit which when added to these three digits and multiplied by itself will produce a number close to 16900963. If I add on 2 then (the square) is too large, so I therefore add a unit to the 411 to produce 4111. The square of this is 16900321. I claim that this is the square nearest to the given number. The number whose root is sought exceeds this square we have found by 642. We say that the root of 16900963 is 4111 and $\frac{642}{8222}$, after doubling 4111.[129] We will now see how adding a unit to 411 cancels out the remaining digits, 7 itself and those digits after it, that is, 7, 9, 6 and 3. We regard each of these as single units [130] and no longer join them as powers of ten [131]. I now add together 7 and 9 and make 16 and say once 8 is 8 and once 2 is 2 leaving 6, and again from the 6 following the 9 I subtract one times 2 leaving 4 and again from the 3 I take one times 1 leaving 2. Joining all these remainders together, that is 6, 4 and 2, I produce 642, which was the excess we found previously. Now these numbers are combined as 642 not 12 [132] since each digit preserves the power of ten from the same place from which it was taken.

Let us do another example in which the root has five digits and let the number be 1690196789.

[129]Note that $4111\frac{642}{8222}^2 \approx 16900963.01$.

[130]Lit. monadic.

[131]Lit. decadically.

[132]i.e. $6 + 4 + 2$.

			1 9	1 9	8							
1	6	9	0	1	9	6	7	8	9			
	4	1	1	1	1	2				2	4	5
		8	2	2	2	4						

I say four times 4 is 16. I write the 4 below the 6 and double it and say once 8 is 8 remainder (from 9 is) 1. I write the one, and I write the remainder 1 which becomes 10 and I subtract from it the one multiplied by itself leaving 9. Double the one and subtract again the 8 from the 9 leaving 1, Adding this to the 1 after the zero and it becomes 11 and I subtract one times 2 from this leaving 9 and further from this we subtract one times itself leaving 8 and then I double the one. Then I take the 8 and all the remaining digits regarding them as single units. Adding the 8 and 9 I get 17. Now I seek a number which when multiplied by itself added to 8 and 2 [and 2][133] can be subtracted from 17, that is, the 8 and 9, either completely from it, with no remainder left, or if there is a remainder, it cancels when combined with the 6 after the 9. If we take 2 times itself combined with the 8 and 2 and 2, then this would require us to subtract 28, but 28 exceeds 8 plus 9 plus 6 and so I take a one. 17 is 8 and 9 and I take 8 once and 2 and 2 and a unit, leaving 4 which I write. Write also the one and double it. See also what number appended[134] to 4111 and the result multiplied by itself will be nearest to the square number we seek. I find this to be 2, since appending it to 4111 produces 41112 and the square of it is 1690196544. The given number exceeds

[133]This has dropped out of the text.
[134]Lit. added.

this by 245 and so we say that the root of the given number is 41112 and 245 over eighty two thousand, two hundred and twenty four. Thus the 2 we have found is subtracted from the 8 and the 2 and the 2 and itself further from the remaining numbers. We will proceed as follows: Add the remaining four with the 6 in turn, I make 10 and I take one from the 10 units. I add it to 10 [135] and it then becomes 19. I take it from it 8 twice, leaving 3. I add this to the following 7 to make 10 and I subtract from it twice 2 and twice 2 leaving 2. Further I take from the 8 in turn the 2, leaving 4 and from the 9, the 2 times itself leaving 5. Combining the digits gives 245. This is the amount by which the number we sought exceeds the nearest square that we found. Notice that the last digit found in the square root, in this case 2, (so that we may explain the procedure carefully), multiplied by itself should be subtracted from the last digit in the first row, and (multiplied) by the penultimate digit, and by the one next in line, third back from the end of the first row, and so on in turn. This is the case not only for numbers whose root has five digits, but also for those with four digits. Observe further that the second digit of the second row is multiplied by itself, and the third digit of that same second row by the first digit of the third and the total is subtracted from the sqaure. This happens when the root has three digits. In the case of four or five digits, and so on, this proceeds even further: The third digit of the second row and the second digit of the third row and by

[135] lit. I make it 10.

itself, are subtracted from the given number, as in the theory we have set down, when to 1 was multiplied by the 2 and also by itself we subtracted from 11. This method, as we have said, is simpler and more general.

It is now finally time for us discuss the method which we invented, and which only slightly departs from the truth. But before doing this, we must show how the[136] preceding (discussion) is not really true.[137] we will demonstrate this both in the case of the numbers and the rows, so that it might become very easily understood.

I will demonstrate this in the case of 24. Let it be required to find the square root of 24. I take the root of its nearest square, that is, of 16. This is 4. I double it and make 8. I then subtract 16 completely from 24 leaving 8. I then claim that the square root of 24 is 4 and[138] eight eighths[139]. But eight eighths is one whole. Now multiplying 4 and eight eighths, or one, by itself, I obtain no longer 24 but 25. Now this, in reality, is the square obtained from five times five, whereas I was expecting to get 24. The 25 is found as follows. I say four times 4 is 16, four times eight eighths, that is one, is 4 and again four times one is 4. One times 1 is 1 and togther this makes 25. The preceding method ignores[140] the last multiplication; the 1 was not multiplied

[136] reading ἡ for ῆ.
[137] lit. able to reach the truth.
[138] The 4 has dropped out of the text. This is understandable since the original would have read 24 4, making it easy for the second 4 to be overlooked by a careless scribe.
[139] The top eight is the remainder while the bottom eight is twice the root.
[140] lit. 'alters'

by the 1 and this seemed to be correct. But if we are going to perform the full multiplication, then one must complete the work by multiplying the 1 by the 1. Observe also that a number following a perfect square becomes larger than itself, but some small amount, when its root is multiplied by itself.[141] The greater the number after the perfect square then the greater the increase and so on until it becomes one unit greater than itself and then increases no further. Then, after this, we reach a number which is by nature a perfect square. Thus, for example, after 16, which is by nature a square, when the square root of 17, calculated according to our method, is multiplied by itself, the result is greater (than 17) by one sixtifourth. For 18 it is one sixteenth, for 19 it is one eighth[142] plus a sixtifourth[143]. For 20 is it is a quarter, for 21 a quarter plus an eighth[144] plus a sixtifourth. For 22 is it a half plus a sixteenth and for 23 is a half plus a quarter plus a sixtifourth. But for 24 it is a full unit. The number after this, 25, is incapable of receiving the same increase but is by nature a perfect square. This assertion manifests itself through all the numbers and is made clear by means of a diagram.

Let two stright lines be drawn at right angles to each other, AB and AC.

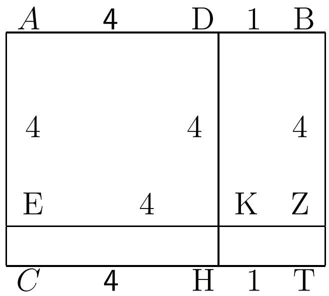
[141] i.e. when we square the square root of a non-square, found according to the above method, then the answer we obtain is somewhat larger than the number we started with.

[142] The Greek has one seventh, which is incorrect. Wäsckhe also writes this. Planudes gives the fractions in the Egyptian manner as sums of unitary fractions.

[143] The excess is $\frac{9}{64} = \frac{1}{8} + \frac{1}{64}$.

[144] The Greek has one seventh, which is incorrect. Wäsckhe also writes this.

Let each be four and eight eighths and let AD (lying) on AB[145] be four units and DB be eight eighths. Likewise on AC, make AE to be four units and EC eight eighths. Construct a square AT[146] and extend EZ parallel to AB and DH parallel to AG.



It is clear that $ADKE$ is a square of side 4, its area[147] is 16. Now since eight eighths is one whole[148], it is clear that each of the sides EG and DB again consists of one whole. Since a whole multiplied by a whole produces a whole, it is clear that each of the parallelograms[149] $CHKE$ and $KDBZ$ are 4 (square) units, totalling 8, for in each, one whole is multiplied by four wholes. It is also the case that $KZTH$ is one whole, for one whole, DB multiplied by one whole , EG[150] makes one whole. The whole square is 16 and 8 and 1 and these added together make 25 which is indeed the case.

[145] Reading $\tau\tilde{\eta}\varsigma$ for $\grave{\epsilon}\tilde{\eta}\varsigma$.

[146] i.e. ABTC. The letter T, which is θ in the Greek, is not actually shown on the accompanying diagram.

[147] lit. 'whole'

[148] Planudes now seems to be using the word $\mu o\tilde{\iota}\rho\alpha$, a 'piece', or 'part', instead of $\mu\acute{o}\nu\alpha\varsigma$, 'unit'.

[149] They are, in fact, rectangles.

[150] Reading $\overline{\epsilon\gamma}$ for $\epsilon\gamma$.

Now let us discuss our method, which precedes as follows. I wish to find, as in the diagram, the square root of 6. I reduce this to seconds and it becomes 21600. I record this and seek the first number for cancelling, I find this to be 1 and say once one is 1. Write this below the 2 leaving 1. Join this to the following 1 and it becomes 11. Double the remainder, which becoming two, I write in the manner (previously) explained. I don't want to annoy you by constantly repeating all the steps. I now look for what number, multiplied by 2 can cancel with 11 and (multiplied) by itself (cancels with) the remainder. I find this to be 4 and say fourtimes two is 8 leaving 3. Write 4 and join the 3 to the following 6 giving 36. I say four times 4 is 16 remainder 20 and write it. Double to 4 to produce 8 and ask what number, multiplied by 2 can cancel with 20, and (multiplied) by 8 (cancels with) the next number and (multiplied) by itself (cancels with) the remainder and we find the number is 6. Now there is also room for 7 or 8 or 9, multiplied by 2, to cancel with 20, but these multiplied by the remaining numbers cannot cancel with these. So I take 6 and say sixtimes 2 is 12 leaving 8. Combine this with the zero to produce 80 and say six times 8 is 48. Subtract this from 80 leaving 32. Combine this with the remaining zero to produce 320. Again I say six times 6 is 36. Subtract 36 from 320 leaving 284. Double the 6 and it becomes 12. I therefore have the nearest square root[151] of 21600 to be singly 146 and doubly 292. The single number squared gives 21316, to which 284 is added

[151]lit. ' the side of the nearest square'

to give 21600˙Therefore, since we have the square root of the nearest square of the number sought, we have need, at this stage, to work from a diagram if we intend to make the lesson easily understood to those engaged in it.

I take the simple root 146 and divide it by 60, and say that this division results in parts[152] and minutes. It is therefore 2 parts and 26 minutes. I lay out the line AB consisting of 2 parts and 26 minutes and I draw the square $ABCD$ on it, making AE and AZ each 2 parts and each of EB anf ZC 26 parts. Draw a line ZT through Z parallel to AB and through EH, through E (parallel to) AC. So $ABCD$ is a complete square (of side) 146 minutes and (area) 21316 seconds[153] since the (side) consists of 2 parts and 26 minutes and the (area) 4 parts, 104 minutes and 676 seconds. For the area of the square and parallelogram $AEKZ$ is 4 units, which arises from AE times AZ, that is, from the parts multiplied by themselves. Also, $CHKZ$ and $KTBE$ have (area) 52, which together gives 104, since EB times EK, that is AZ, and ZC times ZK, that is, AE, thus 26 minutes by 2 parts. Now $KHDT$ has (area) 676 seconds arising from KT, that is EB, times KH, that is ZC, thus 26 minutes multiplied by itself. Thus in total, 4 parts, 104 minutes, resolving these two, and addind to these 676 seconds, we arrive at 21316 seconds, which was our earlier number. So far so good. I reduce the remainder of 294 seconds, after finding the square root, to 17040 tertiaries[154] and take

[152]The Greek word is $\mu o\tilde{\iota}\rho\alpha$ which Planudes uses to describe his units in base 60.

[153]This is rather meaningless,since we cannot have *square seconds* or *square minutes*, nonetheless this is the term Planudes uses and I have kept it in the translation.

[154]i.e. lots of $\frac{1}{60^3}$).

double the root, which is 292 minutes and I seek what number arises when 17040 is divided by 292 and I find 58. Since fifty eight time 292 is 16936 with remainder 104. I extend AB to L and AG to M and make each of BL and GM equal to 58 seconds, since tertiaries divided by minutes produce seconds. I then draw MN equal and parallel to AL and likewise LN equal and parallel to AM and extend EH to Z, BD to O GD to P and ZT to R. I then multiply BL by BT, that is AZ and GM by GH, that is AE. In each case that is 58 seconds by 2 units and in each of (the rectangles) $MXHG$ and $TBLR$ we have 112 seconds. Again I multiply TR, that is BL by TD, that is ZG and HX, that is GM by HD, that is EB. In each case that is 58 seconds by 26 minutes and each of the rectangles $XXHDO$ and $DTRP$[155] has area 1508 tertiaries. The rectangles $MXHG$ and $TRLB$ resolved into tertiaries amd added to $XODH$ and $DPRT$ makes 16936 parts.

There are a further 8 or so pages of text which I have not translated.

[155] reading $\overline{\delta}\rho$ for $\delta\rho$.